The Bernoulli Numbers

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Recall the formulas

$$1^0 + 2^0 + ... + n^0 = n$$

$$1^1 + 2^1 + ... + n^1 = \frac{1}{2}\left(n^2 + n\right)$$

$$1^2 + 2^2 + ... + n^2 = \frac{1}{3}\left(n^3 + \frac{3}{2}n^2 + \frac{1}{2}n\right)$$

$$1^3 + 2^3 + ... + n^3 = \frac{1}{4}\left(n^4 + 2n^3 + n^2\right).$$

$$\text{To prove } 1 + 2 + ... + n = \frac{1}{2}\left(n^2 + n\right):$$

$$S := 1 + 2 + ... + n$$

$$S = n + (n - 1) + ... + 1$$

$$2S = (1 + n) + (2 + n - 1) + ... + (n + 1) = n(n + 1)$$

$$S = n(n + 1)/2$$

To prove $1^2 + 2^2 + ... + n^2 = \frac{1}{3}\left(n^3 + \frac{3}{2}n^2 + \frac{1}{2}n\right)$:

$$R := \{(x, y, z) \in \mathbb{Z}^3 \ : \ 1 \leq x \leq n, \ 1 \leq y \leq n+1, \ 1 \leq z \leq n+1\}.$$

So $|R| = n(n+1)(n+1)$.

We can partition R as $R = R_x \cup R_y \cup R_z$, where

$$R_x = \{(x, y, z) \in R \ : \ x \text{ is maximal } \}$$
$$R_y = \{(x, y, z) \in R \ : \ y \text{ is maximal and } x \text{ is not maximal } \}$$
$$R_z = \{(x, y, z) \in R \ : \ z \text{ is maximal and strictly larger than } x \text{ and } y\}$$

See https://gowers.wordpress.com/2014/11/04/sums-of-kth-powers/

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So $|R| = n(n+1)(n+1)$.

We have:

$$|R_x| = 1^2 + 2^2 + ... + n^2$$

$$|R_y| = (1)(2) + (2)(3) + (3)(4) + ... + (n)(n+1)$$
$$= 1^2 + 2^2 + ... + n^2 + 1 + 2 + ... + n$$

$$|R_z| = 1^2 + 2^2 + ... + n^2$$

See https://gowers.wordpress.com/2014/11/04/sums-of-kth-powers/

We have:

$$\begin{aligned}
|R| &= n(n+1)^2 \\
|R_x| &= 1^2 + 2^2 + ... + n^2 \\
|R_y| &= (1)(2) + (2)(3) + (3)(4) + ... + (n)(n+1) \\
&= 1^2 + 2^2 + ... + n^2 + 1 + 2 + ... + n \\
|R_z| &= 1^2 + 2^2 + ... + n^2
\end{aligned}$$

So $|R| = |R_x| + |R_y| + |R_z|$ gives:

$$n(n+1)^2 = 3(1^2 + 2^2 + ... + n^2) + (1 + 2 + ... + n)$$

$$= 3(1^2 + 2^2 + ... + n^2) + n(n+1)/2$$

and hence $1^2 + 2^2 + ... + n^2 = \dfrac{n(n+1)^2 - n(n+1)/2}{3} = \dfrac{1}{3}\left(n^3 + \dfrac{3}{2}n^2 + \dfrac{1}{2}n\right).$

More generally:

Set $S_k(n) = \sum_{i=1}^{n} i^k$. Then we have

$$S_0(n) = n$$

$$S_1(n) = \frac{1}{2}n^2 - \frac{1}{2}n = \frac{n(n-1)}{2}$$

$$S_2(n) = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n = \frac{n(n-1)(2n-1)}{6}$$

$$S_3(n) = \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2 = \frac{n^2(n-1)^2}{4}$$

$$S_4(n) = \frac{1}{5}n^5 - \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n = \frac{n(n-1)(2n-1)(3n^2-3n-1}{30}$$

$$S_5(n) = \frac{1}{6}n^6 - \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2 = \frac{n^2(2n^2-2n-1)(n-1)^2}{12}.$$

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One can start to see a pattern:

$$S_k(n) = \frac{1}{k+1}n^{k+1} - \frac{1}{2}n^k + \frac{k}{12}n^{k-1} + 0n^{k-2} + ...$$

Here are some Bernoulli numbers:

$$B^0 = 1$$
$$B^1 = 1/2$$
$$B^2 = 1/6$$
$$B^4 = -1/30$$
$$B^6 = 1/42$$
$$B^8 = -1/30$$
$$B^{10} = 5/66$$
$$B^{12} = -691/2730$$
$$B^{14} = 7/6$$
$$B^{16} = -3617/510$$
$$B^{18} = 43867/798$$
$$B^{20} = -174611/330$$
$$B^{22} = 854513/138$$
$$B^{24} = -236364091/2730$$
$$B^{26} = 8553103/6$$
$$B^{28} = -23749461029/870$$
$$B^{30} = 8615841276005/14322$$

Faulhaber's formula for $S_k(n) = \sum_{i=1}^{n} i^k$ states that:

$$S_{k-1}(n) = \frac{1}{k}\left(n^k + \frac{1}{2}\binom{k}{1}n^{k-1} + \frac{1}{6}\binom{k}{2}n^{k-2} + 0\binom{k}{3}n^{k-3} + \frac{-1}{30}\binom{k}{4}n^{k-4} + ...\right)$$

$$= \frac{1}{k}\left(B^0 n^k + B^1\binom{k}{1}n^{k-1} + B^2\binom{k}{2}n^{k-2} + B^3\binom{k}{3}n^{k-3} + B^4\binom{k}{4}n^{k-4} + ...\right).$$

$$S_{k-1}(n) = \frac{1}{k}\left(n^k + \frac{1}{2}\binom{k}{1}n^{k-1} + \frac{1}{6}\binom{k}{2}n^{k-2} + 0\binom{k}{3}n^{k-3} + \frac{-1}{30}\binom{k}{4}n^{k-4} + \dots\right)$$

$$= \frac{1}{k}\left(B^0 n^k + B^1\binom{k}{1}n^{k-1} + B^2\binom{k}{2}n^{k-2} + B^3\binom{k}{3}n^{k-3} + B^4\binom{k}{4}n^{k-4} + \dots\right).$$

Following Conway and Guy, we can derive this formula using a clever notational trick.

Recall the binomial theorem:

$$(a+b)^k = a^k + kba^{k-1} + \binom{k}{2}b^2a^{k-2} + \binom{k}{3}b^3a^{k-3} + \dots + b^k.$$

Let us now agree to interpret expressions like " $(100+B)^k$ " as

$$\text{“ } (100+B)^k \text{ ”} := 100^k + kB^1 100^{k-1} + \binom{k}{2}B^2 100^{k-2} + \dots$$

with $B^0 = 1, B^1 = 1/2, B^2 = 1/6, B^3 = 0, B^4 = -1/30$, etc.

See *The Book of Numbers* by Conway and Guy

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We can now rewrite Faulhaber’s formula as

$$S_{k-1}(n) = \frac{(n+B)^k - B^k}{k}$$

for $S_{k-1}(n) = \sum_{i=1}^{n} i^{k-1}$.

It turns out that Bernoulli numbers satisfy the following identities:

$$B^2 - 2B^1 + 1 = B^2$$

$$B^3 - 3B^2 + 3B^1 - 1 = B^3$$

$$B^4 - 4B^3 + 6B^2 - 4B^1 + 1 = B^4$$

$$B^5 - 5B^4 + 10B^3 - 10B^2 + 5B^1 - 1 = B^5$$

etc.

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$$B^5 - 5B^4 + 10B^3 - 10B^2 + 5B^1 - 1 = B^5$$

etc.

In our notation:

$$(B - 1)^k = B^k$$

for $k \geq 2$.

Now let us compare $(100+B)^k$ with $(99+B)^k$:

$$(100+B)^k = 100^k + kB^1 100^{k-1} + \binom{k}{2} B^2 100^{k-2} + \binom{k}{3} B^3 100^{k-3} + ...$$

$$\begin{aligned}(99+B)^k &= (100+B-1)^k \\ &= 100^k + k(B-1)^1 100^{k-1} + \binom{k}{2}(B-1)^2 100^{k-2} + \binom{k}{3}(B-1)^3 100^{k-3} + ...\end{aligned}$$

Applying the identity, lots of terms cancel:

$$(100+B)^k - (99+B)^k = kB^1 100^{k-1} - k(B-1)^1 100^{k-1} = k100^{k-1}.$$

$$(100+B)^k-(99+B)^k=kB^1 100^{k-1}-k(B-1)^1 100^{k-1}=k100^{k-1}.$$

Similarly, we have

$$
\begin{aligned}
(99+B)^k-(98+B)^k &= k99^{k-1}\\
&\cdots\\
(2+B)^k-(1+B)^k &= k2^{k-1}\\
(1+B)^k-B^k &= k1^{k-1}.
\end{aligned}
$$

Similarly, we have

$$\begin{aligned}(99+B)^k-(98+B)^k &= k99^{k-1}\\ &\dots\\ (2+B)^k-(1+B)^k &= k2^{k-1}\\ (1+B)^k-B^k &= k1^{k-1}.\end{aligned}$$

Adding these up, we get

$$(100+B)^k-B^k = k(1^{k-1}+2^{k-1}+\dots+99^{k-1}+100^{k-1}),$$

which gives Faulhaber's formula!

Some History

Johann Faulhaber

- Born 1580 in Ulm, Germany
- Found a formula for $1^k + 2^k + ... + n^k$, for $k \leq 17$, now known as *Faulhaber's formula*
- The mysterious coefficients in his formula would later become known as the *Bernoulli numbers*

Jacob Bernoulli

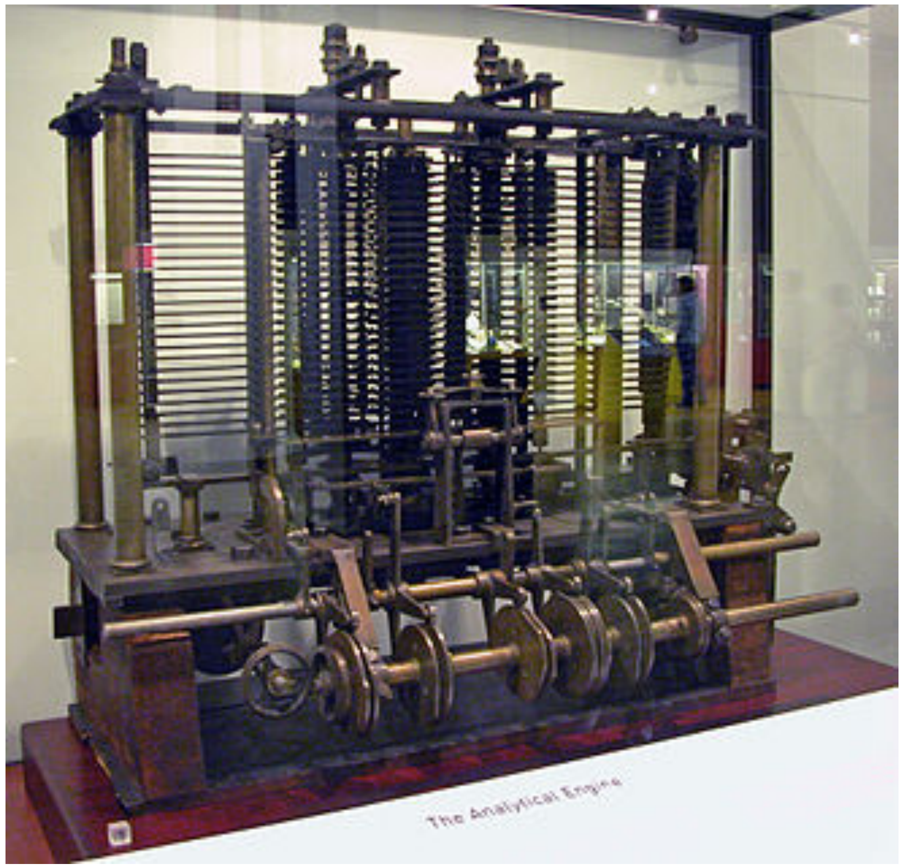
- Born 1655 in Basel, Switzerland
- Gave the first comprehensive treatment of the Bernoulli numbers in *Ars Conjectandi* (1713)

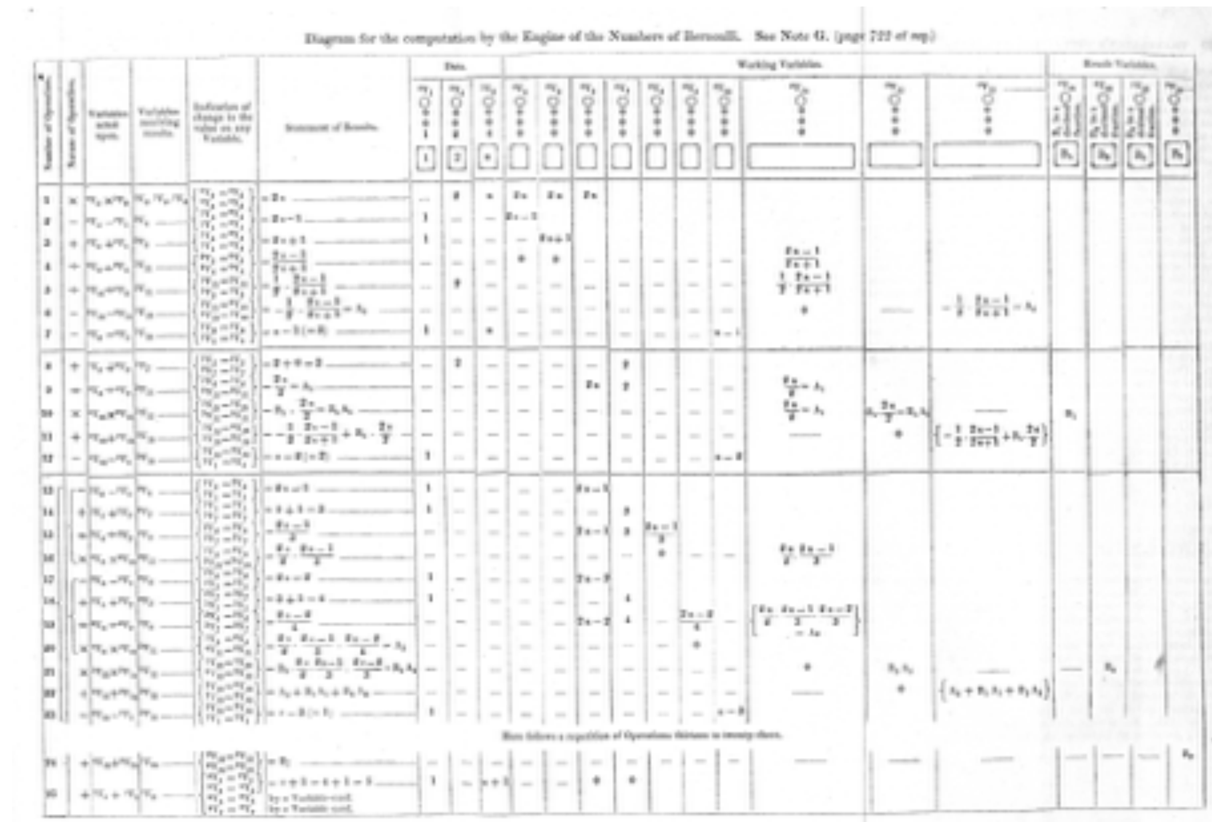
Seki Takakazu

Tabulation of binomial coefficients and Bernoulli numbers from *Katsuyo Sampo* (1712)

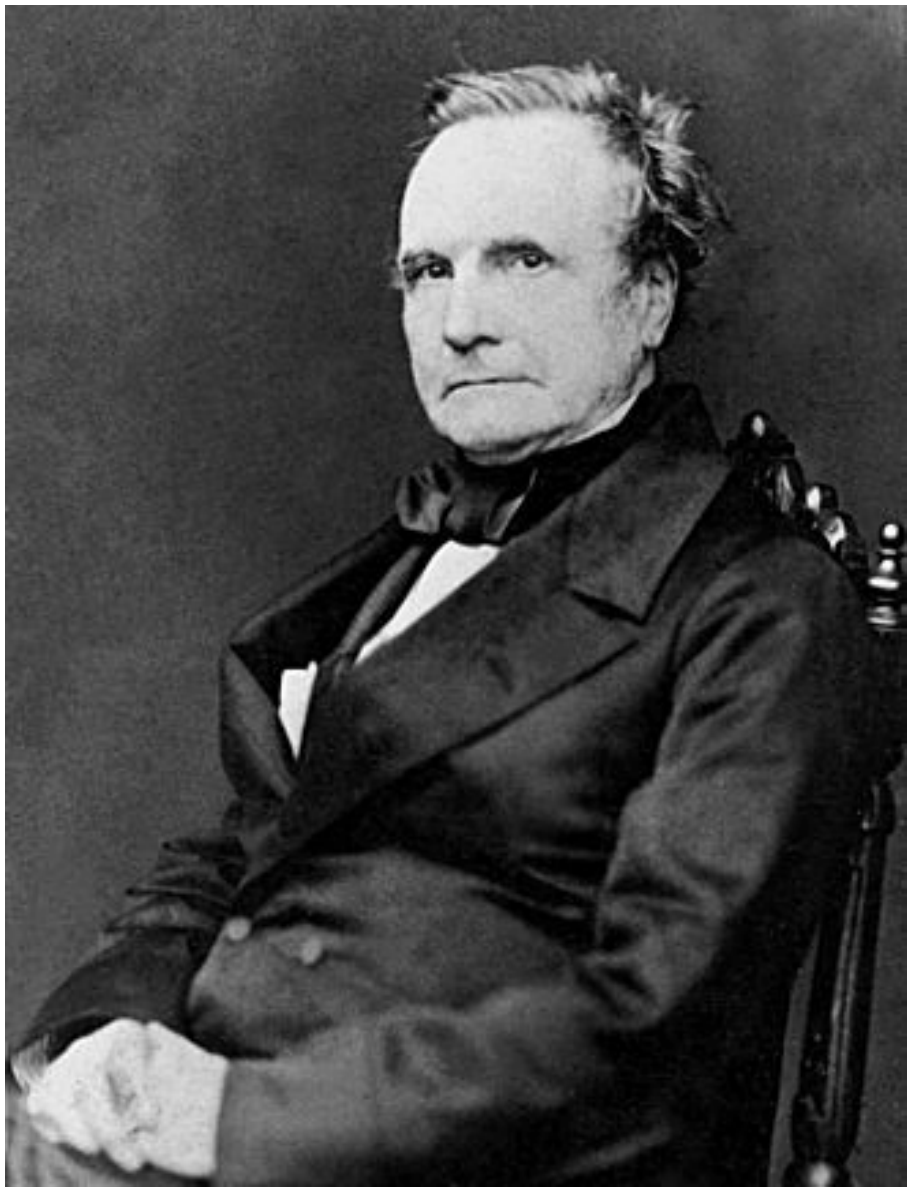
- Born ~1642 in Japan
- Credited with independently discovering the Bernoulli numbers

The Analytical Engine





- Designed by Charles Babbage in 1837
- One of the first designs for a mechanical computer
- Around 1842, the mathematician Ada Lovelace wrote arguably the first computer program, which computes the Bernoulli numbers!



Charles Babbage
b. 1791 in London, England

Ada Lovelace
b. 1815 in London, England

Bernoulli numbers also come up naturally in the Taylor series of trigonometric functions:

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{B^2}{2!}x^2 + \frac{B^4}{4!}x^4 + \frac{B^6}{6!}x^6 + ...$$

$$\tan x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n} (2^{2n} - 1) B^{2n} x^{2n-1}}{(2n)!}$$

$$\csc x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2 (2^{2n-1} - 1) B^{2n} x^{2n-1}}{(2n)!}$$

$$\cot x = \sum_{n=0}^{\infty} \frac{(-1)^{n} 2^{2n} B^{2n} x^{2n-1}}{(2n)!}$$

Denominators of Bernoulli numbers:

The *von Staudt-Clauen Theorem*: the denominator of the 2nth Bernoulli number is the product of all primes p such that p-1 divides 2n.

Ex:

$$B^2 = 1/6; \;\; 2*3 = 6$$

$$B^4 = -1/30; \;\; 2*3*5 = 30$$

$$B^6 = 1/42; \;\; 2*3*7 = 42$$

$$B^8 = -1/30; \;\; 2*3*5 = 30$$

$$B^{10} = 5/66; \;\; 2*3*11 = 66$$

$$B^{12} = -691/2730; \;\; 2*3*5*7*13 = 2730$$

Infinite sum formulas

Euler proved the following formulas:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ... = \frac{1}{6}\pi^2$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + ... = \frac{1}{90}\pi^4$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + ... = \frac{1}{945}\pi^6$$

$$\cdots$$

$$1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \frac{1}{4^{2k}} + ... = \frac{2^{2k-1}}{(2k)!}|B_{2k}|\pi^{2k}.$$

Connections with the Riemann zeta function:

$$\zeta(s) := \sum_{m=0}^{\infty} m^{-s}.$$

- This sum is uniformly convergent on compact subsets of the complex half-space {Re(s) > 0} and hence defines an analytic function of s there
- In fact, $\zeta(s)$ admits an *analytic continuation* to

$$\mathbb{C} \setminus \{1\}.$$

In terms of the Riemann zeta function

$$\zeta(s) := \sum_{m=0}^{\infty} m^{-s},$$

Euler's formulas can be written as

$$\zeta(2k) = \frac{2^{2k-1}}{(2k)!} |B_{2k}| \pi^{2k}.$$

(for k a positive integer)

The Riemann zeta function: $\zeta(s) := \sum_{m=0}^{\infty} m^{-s},$

After extending the domain of the zeta function, one can also compute its values at *negative* integers:

$$\zeta(-n) = -\frac{B^{n+1}}{n+1}.$$

We get rather bizarre formulas:

$$1 + 2 + 3 + ... = \zeta(-1) = -B^2/2 = -1/12$$

$$1^2 + 2^2 + 3^2 + ... = \zeta(-2) = -B^3/3 = 0$$

$$1^3 + 2^3 + 3^3 + ...\zeta(-3) = -B^4/4 = 1/120$$

$$1^4 + 2^4 + 3^4 + ... = \zeta(-4) = -B^5/5 = 0$$

$$1^5 + 2^5 + 3^5 + ... = \zeta(-5) = -B^6/6 = -1/252$$

$$1^6 + 2^6 + 3^6 + ... = \zeta(-6) = -B^7/7 = 0$$

Note that the Bernoulli numbers come up for both negative and positive s values

$$1/1^7 + 1/2^7 + 1/3^7 + ... = \zeta(7) = 1.00834...$$

$$1/1^6 + 1/2^6 + 1/3^6 + ... = \zeta(6) = \frac{2^{2(3)-1}}{6!}|B^6|\pi^6 = \pi^6/945$$

$$1/1^5 + 1/2^5 + 1/3^5 + ... = \zeta(5) = 1.03692...$$

$$1/1^4 + 1/2^4 + 1/3^4 + ... = \zeta(4) = \frac{2^{2(2)-1}}{4!}|B^4|\pi^4 = \pi^4/90$$

$$1/1^3 + 1/2^3 + 1/3^3 + ... = \zeta(3) = 1.20205...$$

$$1/1^2 + 1/2^2 + 1/3^2 + ... = \zeta(2) = \frac{2^{2(1)-1}}{2!}|B^2|\pi^2 = \pi^2/6$$

$$1/1 + 1/2 + 1/3 + ... = \zeta(1) = \infty$$

$$1 + 1 + 1 + ... = \zeta(0) = -B^1 = -1/2$$

$$1 + 2 + 3 + ... = \zeta(-1) = -B^2/2 = -1/12$$

$$1^2 + 2^2 + 3^2 + ... = \zeta(-2) = -B^3/3 = 0$$

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$$1^4 + 2^4 + 3^4 + ... = \zeta(-4) = -B^5/5 = 0$$

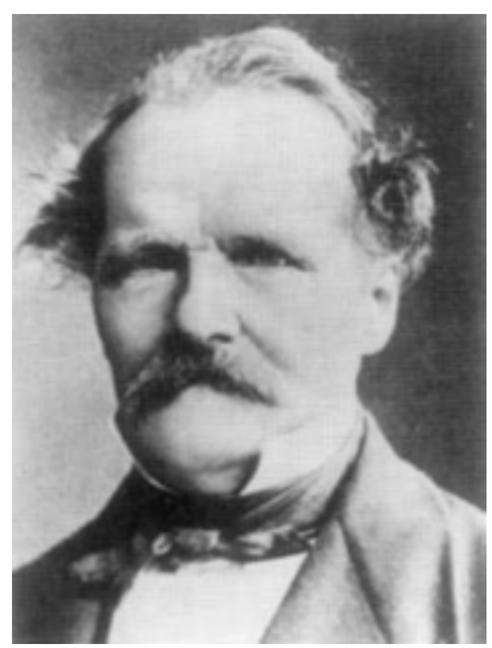
$$1^5 + 2^5 + 3^5 + ... = \zeta(-5) = -B^6/6 = -1/252$$

$$1^6 + 2^6 + 3^6 + ... = \zeta(-6) = -B^7/7 = 0$$

$$1^7 + 2^7 + 3^7 + ... = \zeta(-7) = -B^8/8 = 1/240$$

Regular primes

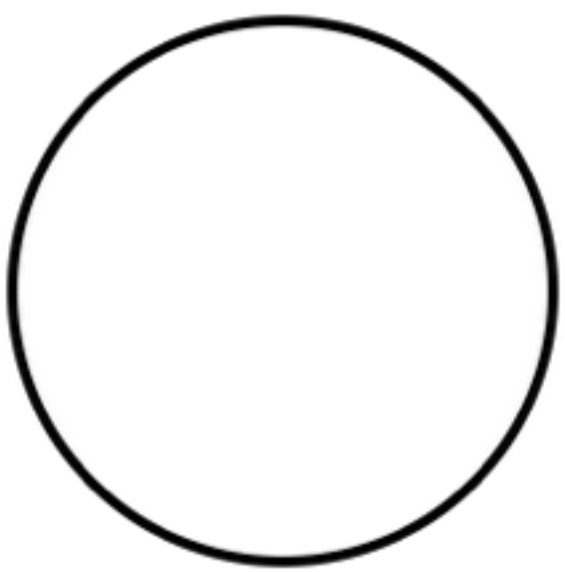
- An prime p is called *regular* if it does not divide the numerator of B^k for k=2,4,6,…,p-3
- It is conjectured that about 60.65% of all primes are regular
- In 1850 Kummer proved Fermat's Last Theorem for regular primes, i.e. a^p+ b^p = c^p has no solutions, provided p is a regular prime. It would be ~150 years before this was



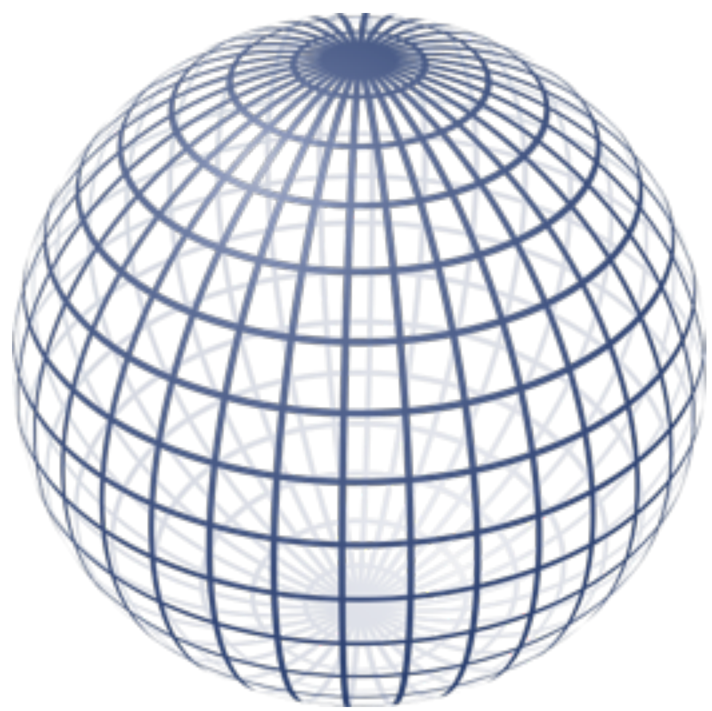
Ernst Kummer
Born 1810 in Sorau, Prussia

Bernoulli numbers in stable homotopy theory

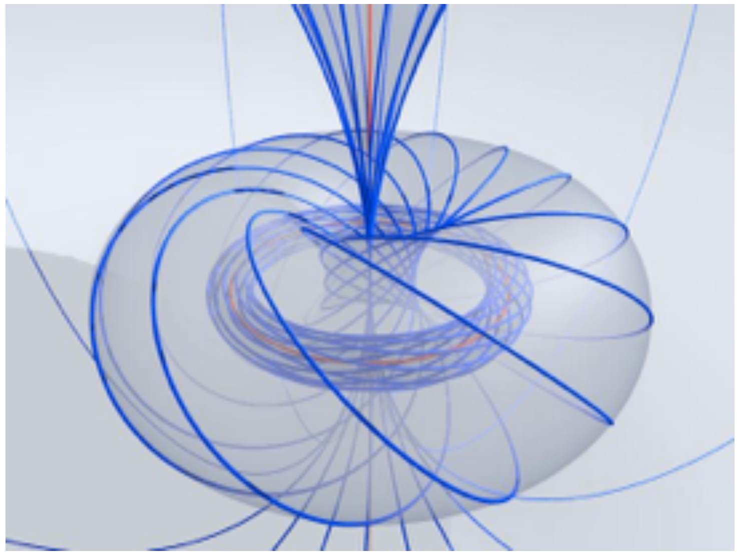
Let S^n denote the n-dimensional sphere.



S^1



S^2



S^3

Let $\pi_i(S^n)$ denote the *ith homotopy group of the n-dimensional sphere:*

$$\pi_i(S^n) := \{\text{continuous maps from } S^i \text{ to } S^n\}/\sim,$$

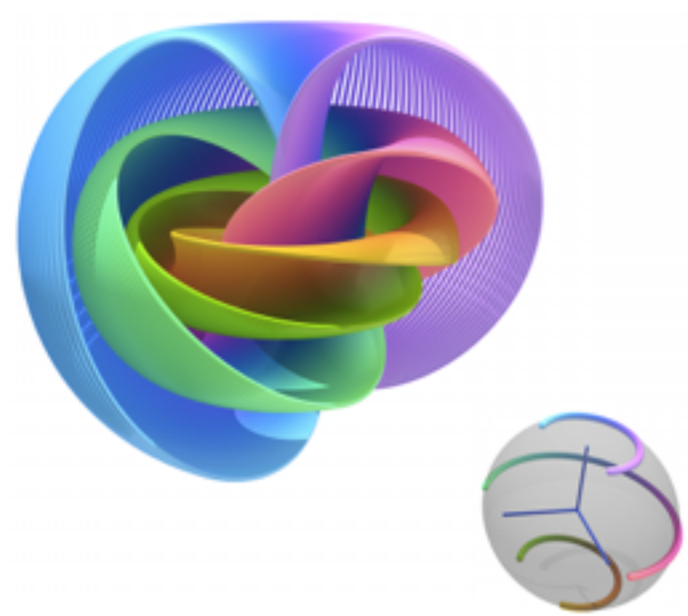
where for maps $f : S^i \to S^n$ and $g : S^i \to S^n$ we have $f \sim g$ if and only if f and be deformed into g through continuous maps

Ex: $\pi_1(S^2)$ has only one element

There are more homotopy groups of spheres than one might expect.

Ex: $\pi_3(S^2) = \mathbb{Z}$,

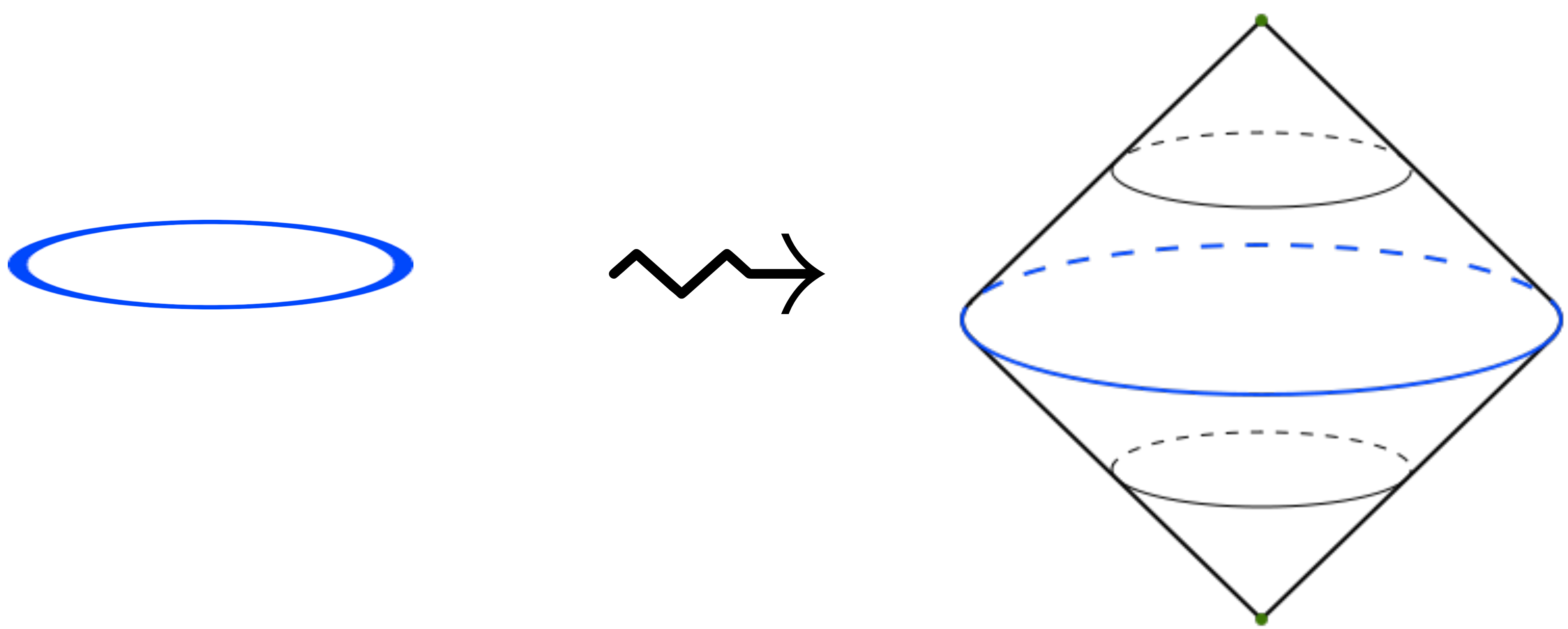
generated by the *Hopf map* $h : S^3 \to S^2$



http://en.wikipedia.org/wiki/Hopf_fibration

Suspension of a topological space

For a topological space X, there is something called the *suspension* of X, denoted ΣX



Ex: $\Sigma S^n = S^{n+1}$

http://en.wikipedia.org/wiki/Suspension_%28topology%29

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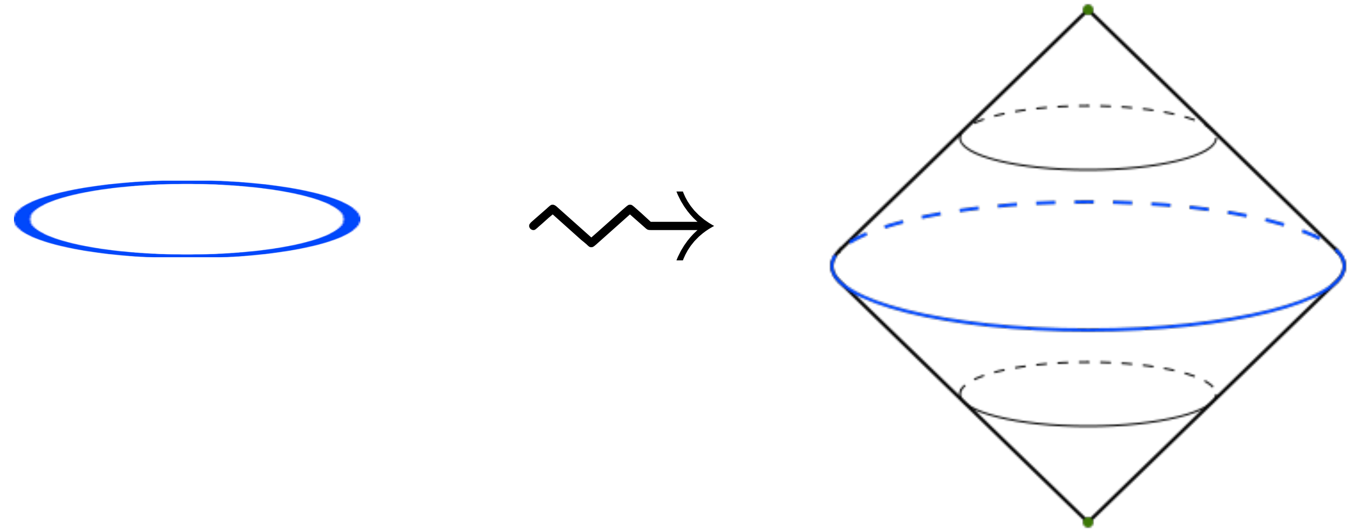
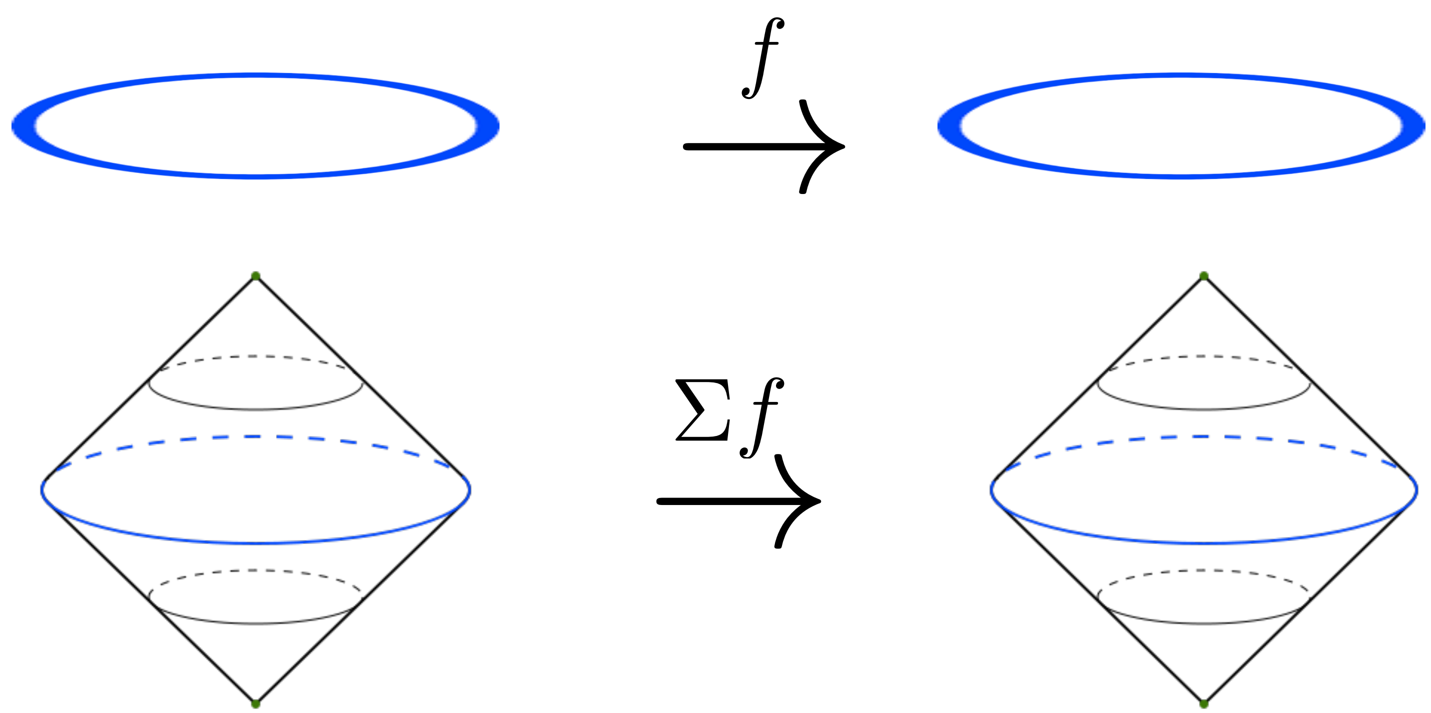


Given a map $f : X \to Y$,

we get a map $\Sigma f : \Sigma X \to \Sigma Y$



This means that there is a map $\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$

The *Freudenthal suspension theorem* states that this is an isomorphism if n > k+1

Let Π_k denote $\pi_{n+k}(S^n)$ for $n >> k$

In general, the field of *stable homotopy theory* studies what happens to topology when you suspend everything in sight a million times.

Computing $\pi_i(S^n)$ is an extremely difficult open problem in topology.

We know the values for small values of i and n:

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^5$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$

http://en.wikipedia.org/wiki/Homotopy_groups_of_spheres#Table_of_stable_homotopy_groups

Computing $\pi_i(S^n)$ is an extremely difficult open problem in topology.

Computing Π_k is in principle easier, but still a very hard open problem.

We know the answer for small values of k:

$n \rightarrow$	0	1	2	3	4	5	6	7
$\pi_{0+n}{}^S$	∞	$\underline{2}$	2	$\underline{8\cdot 3}$	$\cdot$	$\cdot$	2	$\underline{16\cdot 3\cdot 5}$
$\pi_{8+n}{}^S$	$\underline{2}\cdot 2$	$\underline{2}\cdot 2^2$	$2\cdot 3$	$\underline{8\cdot 9\cdot 7}$	$\cdot$	3	2^2	$\underline{32}\cdot 2\cdot \underline{3\cdot 5}$
$\pi_{16+n}{}^S$	$\underline{2}\cdot 2$	$\underline{2}\cdot 2^3$	$8\cdot 2$	$\underline{8}\cdot 2\cdot \underline{3\cdot 11}$	$8\cdot 3$	2^2	$2\cdot 2$	$\underline{16}\cdot 8\cdot 2\cdot \underline{9}\cdot 3\cdot \underline{5\cdot 7\cdot 13}$
$\pi_{24+n}{}^S$	$\underline{2}\cdot 2$	$\underline{2}\cdot 2$	$2^2\cdot 3$	$\underline{8\cdot 3}$	2	3	$2\cdot 3$	$\underline{64}\cdot 2^2\cdot \underline{3\cdot 5\cdot 17}$
$\pi_{32+n}{}^S$	$\underline{2}\cdot 2^3$	$\underline{2}\cdot 2^4$	$4\cdot 2^3$	$\underline{8}\cdot 2^2\cdot \underline{27\cdot 7\cdot 19}$	$2\cdot 3$	$2^2\cdot 3$	$4\cdot 2\cdot 3\cdot 5$	$\underline{16}\cdot 2^5\cdot 3\cdot \underline{3\cdot 25\cdot 11}$
$\pi_{40+n}{}^S$	$\underline{2}\cdot 4\cdot 2^4\cdot 3$	$\underline{2}\cdot 2^4$	$8\cdot 2^2\cdot 3$	$\underline{8\cdot 3\cdot 23}$	8	$16\cdot 2^3\cdot 9\cdot 5$	$2^4\cdot 3$	$\underline{32}\cdot 4\cdot 2^3\cdot \underline{9}\cdot 3\cdot \underline{5\cdot 7\cdot 13}$
$\pi_{48+n}{}^S$	$\underline{2}\cdot 4\cdot 2^3$	$\underline{2}\cdot 2\cdot 3$	$2^3\cdot 3$	$\underline{8}\cdot 4\cdot 2^2\cdot \underline{3}$	$2^3\cdot 3$	2^4	$4\cdot 2$	$\underline{16}\cdot 3\cdot \underline{3\cdot 5\cdot 29}$
$\pi_{56+n}{}^S$	$\underline{2}\cdot 2$	$\underline{2}\cdot 2^3$	2^2	$\underline{8}\cdot 2^2\cdot \underline{9\cdot 7\cdot 11\cdot 31}$	4	$\cdot$	$4\cdot 2^2\cdot 3$	$\underline{128}\cdot 2^3\cdot \underline{3\cdot 5\cdot 17}$

http://en.wikipedia.org/wiki/Homotopy_groups_of_spheres#Table_of_stable_homotopy_groups

Bernoulli numbers enter the picture via the so-called *J homomorphism*

Let O(n) denote the *orthogonal group* consisting of all nxn matrices A such that $A^TA = AA^T = I_n$

O(n) is a *Lie group*. Roughly speaking this means that:

- It is a group, i.e. we can multiply and invert elements
- It is a topological space, i.e there is a natural notion of two elements being “close” to each other

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Since O(n) is a topological space, it makes sense to talk about its homotopy groups

It turns out that $\pi_k(O(n))$ is independent of k for n >> k

Bott computed $\pi_k(O(\infty))$, and remarkably the answer is 8-periodic!

Bott periodicity:

$$\begin{aligned}
\pi_0(O(\infty)) &\cong \mathbb{Z}/2 \\
\pi_1(O(\infty)) &\cong \mathbb{Z}/2 \\
\pi_2(O(\infty)) &\cong 0 \\
\pi_3(O(\infty)) &\cong \mathbb{Z} \\
\pi_4(O(\infty)) &\cong 0 \\
\pi_5(O(\infty)) &\cong 0 \\
\pi_6(O(\infty)) &\cong 0 \\
\pi_7(O(\infty)) &\cong \mathbb{Z} \\
\pi_8(O(\infty)) &\cong \pi_0 \\
\pi_9(O(\infty)) &\cong \pi_1 \\
& etc.
\end{aligned}$$

Raoul Bott
b. 1923 in Budapest, Hungary

There is a homomorphism

$$J : \pi_i(O(n)) \to \pi_{i+n}(S^n)$$

Taking n to be rather large, we get a subgroup $\mathrm{Im}(J) \subset \Pi_i$

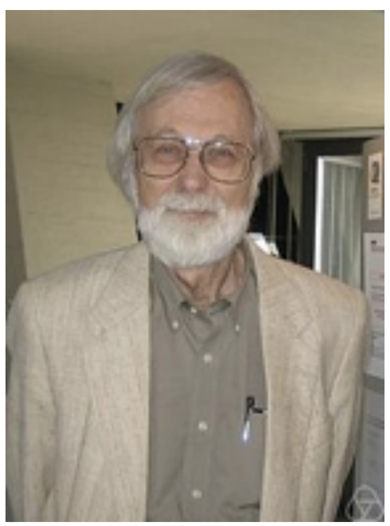
$\mathrm{Im}(J)$ consists of classes of maps $f : S^a \to S^b$ such that $f^{-1}(p)$ for a random point $p \in S^b$ is a $(a-b)$-dimensional sphere

$$|\mathrm{Im}(J)| = 2 \cdot \text{ denominator } \left(\frac{B^{2k}}{2k}\right)$$

for $\mathrm{Im}(J) \subset \Pi_{4k-1}$

Bernoulli also come up in formulas for the numbers of "exotic spheres" in dimension n

Let Θ_n denote the group of n-dimensional *exotic spheres*



Milnor and Kervaire (1962) showed:

$$|\Theta_{4k-1}| = R(k) \cdot |\Pi_{4k-1}| \cdot B^{2k}/2k$$

where

$$R(k) := 2^{2k-3}(2^{2k-1} - 1) \qquad \text{if } k \text{ is even}$$
$$R(k) := 2^{2k-2}(2^{2k-1} - 1) \qquad \text{if } k \text{ is odd}$$

Thanks for listening!